An elementary derivation of the uncertainty principle

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1 Definitions

Let H be a complete vector space over the complex numbers $\mathbb{C}$. We define an *inner product* $\langle \text{-}|\text{-}\rangle$ on H as a function from $H \times H$ to $\mathbb{C}$ satisfying the following properties for any ϕ, ψ and ρ in H, and for any $\alpha \in \mathbb{C}$.

1. $\langle \phi|\psi\rangle = \langle \psi|\phi\rangle^*$ (conjugate symmetry)
2. $\langle \phi|\alpha\psi + \rho\rangle = \alpha \langle \phi|\psi\rangle + \langle \phi|\rho\rangle$ (linearity in second argument)
3. $\langle \phi|\phi\rangle \geq 0$ and $\langle \phi|\phi\rangle = 0$ if and only if $\phi = 0$ (positive definiteness)

where $\langle \text{-}|\text{-}\rangle^*$ denotes complex conjugation.[1]

An *operator* is a map from H to itself. For an operator A and a state vector ϕ, the *expectation* of A is given by $\langle A\rangle \coloneqq \langle \phi|A\phi\rangle \eqqcolon \langle \phi|A|\phi\rangle$. We define the *adjoint* (or *Hermitian conjugate*) of A, denoted as $A^\dagger$ by its action on the inner product:

$$\langle \phi|A^\dagger|\psi\rangle \coloneqq \langle \psi|A|\phi\rangle^*$$

If $A^\dagger = A$, we say that A is *self-adjoint* (or *Hermitian*), and if $A^\dagger = -A$, we call A *skew-adjoint* (or *anti-Hermitian*).

The uncertainty ΔA is given by the standard deviation of the distribution of A, that is, it is the positive square root of $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$; this captures the intuition of a measure of how the measurements are spread around the expectation. Finally, the *commutator* of two observables A and B is given by $[A, B] = AB - BA$.

We now have enough structure to derive the simple form of the uncertainty principle.

2 Derivation

We first prove some elementary results about expectation values and commutators.

Lemma 1. *The expectation of any skew-adjoint operator is purely imaginary.*

[1] For the state vectors in quantum mechanics, the inner product is defined as a certain integral satisfying these three properties, we will, however, only need the defining properties of an inner product rather than its explicit form.

Proof. Suppose C is a skew-adjoint operator, that is, $C^\dagger = -C$. We thus have:

$$\begin{aligned}\langle C\rangle &= \langle\phi|C|\phi\rangle \\ &= -\langle\phi|C^\dagger|\phi\rangle \\ &= -\langle\phi|C|\phi\rangle^* \\ &= -\langle C\rangle^*\end{aligned}$$

Thus $\langle C\rangle^* = -\langle C\rangle$, which is only true if $\langle C\rangle$ is purely imaginary. □

By almost an identical argument, differing only by the minus sign, we get:

Lemma 2. *The expectation of any self-adjoint operator is a real number.*

Lemma 3. *The commutator of self-adjoint operators is skew-adjoint.*

Proof. Suppose A and B are self-adjoint. We immediately get:

$$\begin{aligned}[A,B]^\dagger &= (AB - BA)^\dagger \\ &= B^\dagger A^\dagger - A^\dagger B^\dagger && \text{using linearity and that } (AB)^\dagger = B^\dagger A^\dagger \\ &= BA - AB && \text{using the assumption that } A \text{ and } B \text{ are self-adjoint} \\ &= -[A,B]\end{aligned}$$

□

Hence, combining Lemma 4 with Lemma 2 we obtain the following result; the expectation of the commutator of any self-adjoint operators is purely imaginary.

The last result we will need is that the operator $\bar{A} := A - \langle A\rangle$ has the same uncertainty as A. This follows by noting that the expectation of $\bar{A}$ is zero, and hence the square of the uncertainty is simply:

$$\begin{aligned}(\Delta\bar{A})^2 &= \langle\bar{A}^2\rangle \\ &= \langle(A^2 + \langle A\rangle^2 - 2A\langle A\rangle)\rangle \\ &= \langle A^2\rangle - \langle A\rangle^2 \\ &= (\Delta A)^2\end{aligned}$$

This means whatever is true for the uncertainty of $\bar{A}$ is also true for the uncertainty of A.

We now have all the ingredients to state and prove the uncertainty principle.

Proposition 4. *For any self-adjoint operators A and B, we have*

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[A,B]\rangle\right|.$$

Proof. Let $\bar{A} := A - \langle A\rangle$ and $\bar{B} := B - \langle B\rangle$. Hence we have $(\Delta A)^2 = (\Delta\bar{A})^2 = \langle\bar{A}^2\rangle$ and $(\Delta B)^2 = (\Delta\bar{B})^2 = \langle\bar{B}^2\rangle$.

Next, if ϕ is the vector with respect to which the expectations are defined, define a new vector $\psi = (\bar{A}t + \bar{B}i)\phi$ where t is a real variable and i is the imaginary unit. Now consider $\langle\psi|\psi\rangle$.

$$
\begin{aligned}
\langle\psi|\psi\rangle &= \langle(\bar{A}t + \bar{B}i)\phi|(\bar{A}t + \bar{B}i)\phi\rangle \\
&= \langle\phi|(\bar{A}t + \bar{B}i)^\dagger|(\bar{A}t + \bar{B}i)\phi\rangle^* \\
&= \langle\phi|(\bar{A}t - \bar{B}i)(\bar{A}t + \bar{B}i)|\phi\rangle \\
&= \langle\phi|\bar{A}^2t^2 + \bar{B}^2 + it\bar{A}\bar{B} - it\bar{B}\bar{A}|\phi\rangle \\
&= \langle\phi|\bar{A}^2t^2 + \bar{B}^2 + it[\bar{A}, \bar{B}]|\phi\rangle \\
&= \langle\bar{A}^2\rangle t^2 + \langle\bar{B}^2\rangle + it\,\langle[\bar{A}, \bar{B}]\rangle \\
&= (\Delta A)^2t^2 + (\Delta B)^2 + it\,\langle[A, B]\rangle \qquad \text{using that } [\bar{A}, \bar{B}] = [A, B]
\end{aligned}
$$

Note that for all $t \in \mathbb{R}$ this is indeed a real number, since $(\Delta A)^2$ and $(\Delta B)^2$ are real being defined via expectations of self-adjoint operators, and $i\,\langle[A, B]\rangle$ is a product of two purely imaginary numbers and thus real. Moreover, by the definition of an inner product we must have $\langle\psi|\psi\rangle \geq 0$ for all t. This is therefore a quadratic equation in t with at most one root, which means its discriminant must be less than or equal to zero:

$$D = |\langle[A, B]\rangle|^2 - 4(\Delta A)^2(\Delta B)^2 \leq 0,$$

rearranging this we get:

$$\Delta A \Delta B \geq \frac{1}{2}\,|\langle[A, B]\rangle|\,.$$

□

It is interesting to note that up to this point we have not used any physics, the uncertainty relation followed purely as a property of self-adjoint operators. The physics comes in with the fact that the vectors $(\phi,\ \psi,\ ...)$ represent the quantum states and that each observable quantity is associated with a self-adjoint operator $(A,\ B,\ ...)$. This is somewhat plausible, since the self-adjoint operators are guaranteed to have real-valued expectations, a desirable quality of a measurable quantity. There is, therefore, an uncertainty relation between any two non-commuting observables, with the lower bound of the product of the uncertainties being equal to half of the magnitude of the commutator.[2] Thus given the commutation relation between position and momentum $[X, P] = i\hbar$ (another bit of physics), we immediately get the most famous uncertainty relation.

[2] The inequality is, of course, also true, though not very interesting, when the observables commute, as it simply says that the product of two nonnegative numbers is nonnegative.